UNIFICATION OF SPINS AND CHARGES IN GRASSMANN SPACE ENABLES UNIFICATION OF ALL INTERACTIONS [1]

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Abstract

In a space of d Grassmann coordinates two types of generators of Lorentz transformations can be defined, one of spinorial and the other of vectorial character. Both kinds of operators appear as linear operators in Grassmann space, definig the fundamental and the adjoint representations of the group $SO(1, d-1)$, respectively. The eigenvalues of commuting operators belonging to the subgroup $(SO(1,4))$ can be identified with spins of either fermionic or bosonic fields, while the operators belonging to subgroups of $SO(d-5) \supset SU(3) \times SU(2) \times U(1)$, determine the Yang-Mills charges. The theory offers unification of all the internal degrees of freedom of fermionic and bosonic fields - spins and all Yang-Mills charges. When accordingly all interactions are unified, Yang-Mills fields appear as part of the gravitational field. The theory suggests that elementary particles are either in the fundamental representations with respect to the groups determining the spin and the charges, or they are in the adjoint representations with respect to the groups, which determine the spin and the charges.

INTRODUCTION

What today is accepted as elementary particles and fields are either *fermionic fields with the internal degrees of freedom - spins and all charges - in the fundamental representations with respect to the groups $SO(1,3), U(1), SU(2), SU(3)$ or bosonic fields with the internal degrees of freedom - spins and charges - in the adjoint (regular) representations with respect to the same groups.* Fermions are the Lorentz spinors, forming the isospin doublets or singlets, the colour triplets or singlets and the charge singlets. Bosons are the Lorentz scalars or vectors, forming the isospin singlets or triplets, the colour singlets or octets and the charge singlets. There exist no known fermions with charges in the adjoint representations and no known bosons with charges in the fundamental

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representations. The Higgs scalar appears in the Standard model as an isospin doublet, but his existence is not yet prooved.

In this talk I am proposing the approach which unifies spins and charges requiring that (elementary) fermions are in the fundamental representations with respect to all gauge groups and (elementary) bosons are in the adjoint representations with respect to all gauge groups. The space has d commuting and d anticommuting (Grassmann) coordinates. All the internal degrees of freedom, spins and charges, are described by the generators of the Lorentz transformations in Grassmann space. All gauge fields - gravitational as well as Yang-Mills including electrodynamics - are defined by (super)vielbeins.

In Grassmann space there are two types of generators of Lorentz transformations and translations: one is of spinorial character and determines properties of fermions, the other is of vectorial character and determines properties of bosons. Both types of generators are *linear differential operators in Grassmann space.* Their representations can be expressed as monomials of Grassmann coordinates θ^a. If $d \geq 15$ the generators of the subgroup $SO(1,4)$ of the group $SO(1,14)$ determine spins of fields, while generators of the subgroups $SU(3), SU(2), U(1)$ determine their charges.

The Lagrange function describing a particle on a supergeodesics, leads to the momentum of the particle in Grassmann space which is proportional to the Grassmann coordinate. This brings the Clifford algebra and the spinorial degrees of freedom into the theory. The supervielbeins, transforming the geodesics from the freely falling to the external coordinate system, carry the vectorial as well as the spinorial degrees of freedom. All fields, the fermionic and the bosonic fields, depend on ordinary and Grassmann coordinates, the later determine spins and charges of fields.

The Yang-Mills fields appear as the contribution of gravity through spin connections and not through vielbeins as in the Kaluza-Klein theories. Because of that and because the generators of the Lorentz transformations in Grassmann space rather then momentum in ordinary space determine charges of fields, the Planck mass of charged particles and the transformation inconsistencies of gauge fields in Kaluza-Klein theories do not appear.

More about this approach can be found in Refs.[1, 2].

COORDINATE GRASSMANN SPACE AND LINEAR OPERATORS

In this section we briefly repeat a few definitions concerning a d-dimensional Grassmann space, linear Grassmann space spanned over the coordinate space, linear operators defined in this space and the Lie algebra of generators of the Lorentz transformations [2, 3].

Coordinate space with Grassmann character

We define a d-dimensional Grassmann space of real anticommuting coordinates $\{\theta^a\}$, $a = 0, 1, 2, 3, 5, 6, ..., d$, satisfying the anticommutation relations

$$\theta^a \theta^b + \theta^b \theta^a := \{\theta^a, \theta^b\} = 0, \tag{2.1}$$

called the Grassmann algebra [2, 3]. The metric tensor $\eta_{ab} = diag(1, -1, -1, -1, ..., -1)$ lowers the indices of a vector $\{\theta^a\} = \{\theta^0, \theta^1, ..., \theta^d\}, \theta_a = \eta_{ab}\theta^b$. Linear transformation actions on vectors $(\alpha\theta^a + \beta x^a)$

$$(\alpha\acute{\theta}^a + \beta\acute{x}^a) = L^a{}_b(\alpha\theta^b + \beta x^b), \tag{2.2}$$

which leave forms

$$(\alpha\theta^a + \beta x^a)(\alpha\theta^b + \beta x^b)\eta_{ab} \tag{2.3}$$

invariant, are called the Lorentz transformations. Here $(\alpha\theta^a + \beta x^a)$ is a vector of d anticommuting components (Eq.(2.1)) and d commuting $(x^a x^b - x^b x^a = 0)$ components, and α and β are two complex numbers. The requirement that forms (2.3) are scalars with respect to the linear transformations (2.2), leads to the equations

$$L^a{}_c L^b{}_d \eta_{ab} = \eta_{cd}. \tag{2.4}$$

Linear vector space

A linear space spanned over a Grassmann coordinate space of d coordinates has the dimension 2^d. If monomials $\theta^{\alpha_1}\theta^{\alpha_2}....\theta^{\alpha_n}$, are taken as a set of basic vectors with $\alpha_i \neq \alpha_j$, half of the vectors have an even (those with an even n) and half of the vectors have an odd (those with an odd n) Grassmann character. Any vector in this space may be represented as a linear superposition of monomials

$$f(\theta) = \alpha_0 + \sum_{i=1}^{d} \alpha_{a_1 a_2 .. a_i}\theta^{a_1}\theta^{a_2}....\theta^{a_i}, \quad a_k < a_{k+1}, \tag{2.5}$$

where constants $\alpha_0, \alpha_{a_1 a_2 .. a_i}$ are complex numbers.

Linear operators

In Grassmann space the left derivatives have to be distinguished from the right derivatives, due to the anticommuting nature of the coordinates [2, 3]. We shall make use of left derivatives $\overrightarrow{\partial^\theta}_a := \frac{\overrightarrow{\partial}}{\partial\theta^a}$, $\overrightarrow{\partial^\theta}^a := \eta^{ab}\overrightarrow{\partial^\theta}_b$, on vectors of the linear space of monomials $f(\theta)$, defined as follows:

$$\overrightarrow{\partial^\theta}_a \theta^b f(\theta) = \delta^b{}_a f(\theta) - \theta^b \overrightarrow{\partial^\theta}_a f(\theta), \tag{2.6}$$

$$\overrightarrow{\partial^\theta}_a \alpha f(\theta) = (-1)^{n_{a\partial}}\alpha\overrightarrow{\partial^\theta}_a f(\theta).$$

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Here α is a constant of either commuting ($\alpha\theta^a - \theta^a\alpha = 0$) or anticommuting ($\alpha\theta^a + \theta^a\alpha = 0$) character, and $n_{a\partial}$ is defined as follows

$$n_{AB} = \left\{ \begin{array}{ll} +1, & if\ A\ and\ B\ have\ Grassmann\ odd\ character \\ 0, & otherwise \end{array} \right\}$$

We define the following linear operators [1, 2]

$$p^{\theta}{}_a := -i\overrightarrow{\partial^{\theta}}{}_a, \ \ \tilde{a}^a := i(p^{\theta a} - i\theta^a), \ \ \tilde{\tilde{a}}^a := -(p^{\theta a} + i\theta^a). \tag{2.7}$$

According to the inner product defined in the next subsection, the operators $\tilde{a}^a$ and $\tilde{\tilde{a}}^a$ are either hermitian or antihermitian operators

$$\tilde{a}^{a+} = -\eta^{aa}\tilde{a}^a, \ \ \tilde{\tilde{a}}^{a+} = -\eta^{aa}\tilde{\tilde{a}}^a. \tag{2.7a}$$

We define the generalized commutation relations [1, 2] (we shall show later that they follow from the corresponding Poisson brackets):

$$\{A, B\} := AB - (-1)^{n_{AB}}BA, \tag{2.8}$$

fulfilling the equations

$$\{A, B\} = (-1)^{n_{AB}+1}\{B, A\}, \tag{2.9a}$$

$$\{A, BC\} = \{A, B\}C + (-1)^{n_{AB}}B\{A, C\}, \tag{2.9b}$$

$$\{AB, C\} = A\{B, C\} + (-1)^{n_{BC}}\{A, C\}B, \tag{2.9c}$$

$$(-1)^{n_{AC}}\{A, \{B, C\}\} + (-1)^{n_{CB}}\{C, \{A, B\}\} + (-1)^{n_{BA}}\{B, \{C, A\}\} = 0. \tag{2.9d}$$

We find

$$\{p^{\theta a}, p^{\theta b}\} = 0 = \{\theta^a, \theta^b\}, \tag{2.10a}$$

$$\{p^{\theta a}, \theta^b\} = -i\eta^{ab}, \tag{2.10b}$$

$$\{\tilde{a}^a, \tilde{a}^b\} = 2\eta^{ab} = \{\tilde{\tilde{a}}^a, \tilde{\tilde{a}}^b\}, \tag{2.10c}$$

$$\{\tilde{a}^a, \tilde{\tilde{a}}^b\} = 0. \tag{2.10d}$$

We see that θ^a and $p^{\theta a}$ form a Grassmann odd Heisenberg algebra, while $\tilde{a}^a$ and $\tilde{\tilde{a}}^a$ form the Clifford algebra.

We define the projectors

$$P_\pm = \frac{1}{2}(1 \pm \sqrt{(-)^{\tilde{\Upsilon}\tilde{\tilde{\Upsilon}}}}\,\tilde{\Upsilon}\tilde{\tilde{\Upsilon}}), \quad (P_\pm)^2 = P_\pm, \qquad (2.11)$$

where $\tilde{\Upsilon}$ and $\tilde{\tilde{\Upsilon}}$ are the two operators defined for any dimension d as follows

$$\tilde{\Upsilon} = i^\alpha \prod_{a=0,1,2,3,5,..,d} \tilde{a}^a \sqrt{\eta^{aa}}, \qquad (2.11a)$$

$$\tilde{\tilde{\Upsilon}} = i^\alpha \prod_{a=0,1,2,3,5,..,d} \tilde{\tilde{a}}^a \sqrt{\eta^{aa}}, \qquad (2.11b)$$

with α equal either to $d/2$ or to $(d-1)/2$ for even and odd dimension d of the space, respectively.

It can be checked that $(\tilde{\Upsilon})^2 = 1 = (\tilde{\tilde{\Upsilon}})^2$.

The projectors $P_\pm$ project out of any monomials of Eq.(2.5) the Grassmann odd and the Grassmann even part of the monomial, respecticely.

We find that for odd d the operators $\tilde{\Upsilon}$ and $\tilde{\tilde{\Upsilon}}$ coincide (up to $\pm i$ or ± 1) with $\tilde{\Gamma}$ and $\tilde{\tilde{\Gamma}}$ of Eq.(2.17), respectively.

Lie algebra of generators of Lorentz transformations

We define two kinds of operators [2]. The first ones are binomials of operators forming the Grassmann odd Heisenberg algebra

$$S^{ab} := (\theta^a p^{\theta b} - \theta^b p^{\theta a}). \qquad (2.12a)$$

The second kind are binomials of operators forming the Clifford algebra

$$\tilde{S}^{ab} := -\frac{i}{4}[\tilde{a}^a, \tilde{a}^b], \quad \tilde{\tilde{S}}^{ab} := -\frac{i}{4}[\tilde{\tilde{a}}^a, \tilde{\tilde{a}}^b], \qquad (2.12b)$$

with $[A, B] := AB - BA$.

Either S^{ab} or $\tilde{S}^{ab}$ or $\tilde{\tilde{S}}^{ab}$ fulfil the Lie algebra of the Lorentz group $SO(1, d-1)$ in the d-dimensional Grassmann space :

$$\{M^{ab}, M^{cd}\} = -i(M^{ad}\eta^{bc} + M^{bc}\eta^{ab} - M^{ac}\eta^{bd} - M^{bd}\eta^{ac}) \qquad (2.13)$$

with M^{ab} equal either to S^{ab} or to $\tilde{S}^{ab}$ or to $\tilde{\tilde{S}}^{ab}$ and $M^{ab} = -M^{ba}$.

We see that

$$S^{ab} = \tilde{S}^{ab} + \tilde{\tilde{S}}^{ab}, \quad \{\tilde{S}^{ab}, \tilde{\tilde{S}}^{cd}\} = 0 = \{\tilde{S}^{ab}, \tilde{\tilde{a}}^c\} = \{\tilde{a}^a, \tilde{\tilde{S}}^{bc}\}. \qquad (2.14)$$

By solving the eigenvalue problem (see Sect. 2.6) we find that operators $\tilde{S}^{ab}$, as well as the operators $\tilde{\tilde{S}}^{ab}$, define the fundamental or the spinorial representations of the Lorentz group, while $S^{ab} = \tilde{S}^{ab} + \tilde{\tilde{S}}^{ab}$ define the regular or the adjoint or the vectorial representations of the Lorentz group $SO(1, d-1)$.

Group elements are in any of the three cases defined by:

$$\mathcal{U}(\omega) = e^{\frac{i}{2}\omega_{ab}M^{ab}}, \tag{2.15}$$

where ω_{ab} are the parameters of the group.

Linear transformations, defined in Eq.(2.2), can then be written in terms of group elements as follows

$$\acute{\theta}^a = L^a{}_b\theta^b = e^{-\frac{i}{2}\omega_{cd}S^{cd}}\theta^a e^{\frac{i}{2}\omega_{cd}S^{cd}}.$$

By using Eqs.(2.9) and (2.13) it can be proved for any d , that M^2 is the invariant of the Lorentz group

$$\{M^2, M^{cd}\} = 0, \quad M^2 = \frac{1}{2}M^{ab}M_{ab}, \tag{2.16}$$

and that for d=2n we can find the additional invariant Γ

$$\{\Gamma, M^{cd}\} = 0, \quad \Gamma = \frac{i(-2i)^n}{(2n)!}\epsilon_{a_1a_2...a_{2n}}M^{a_1a_2}....M^{a_{2n-1}a_{2n}}, \tag{2.17}$$

where $\epsilon_{a_1a_2...a_{2n}}$ is the totally antisymmetric tensor with $2n$ indices and with $\epsilon_{123...2n} = 1$. This means that M^2 and Γ are for $d = 2n$ the two invariants or Casimir operators of the group $SO(d)$ (or $SO(1, d-1)$, the two algebras differ only in the definition of the metric η^{ab}). For $d = 2n+1$ the second invariant cannot be defined.(It can be checked that $\tilde{\Upsilon}$ and $\tilde{\tilde{\Upsilon}}$ of eqs.(2.11) are the two invariants for the spinorial case for any d. For even d they coincide with $\tilde{\Gamma}$ and $\tilde{\tilde{\Gamma}}$, respectively, while for odd d the eigenvectors of these two operators are superpositions of Grassmann odd and Grassmann even monomials.)

While the invariant M^2 is trivial in the case when M^{ab} has spinorial character, since $(\tilde{S}^{ab})^2 = \frac{1}{4}\eta^{aa}\eta^{bb} = (\tilde{\tilde{S}}^{ab})^2$ and therefore M^2 is equal in both cases to the number $\frac{1}{2}\tilde{S}^{ab}\tilde{S}_{ab} = \frac{1}{2}\tilde{\tilde{S}}^{ab}\tilde{\tilde{S}}_{ab} = d(d-1)\frac{1}{8}$, it is a nontrivial differential operator in Grassmann space if M^{ab} have vectorial character ($M^{ab} = S^{ab}$). The invariant of Eq.(2.17) is always a nontrivial operator.

Integrals on Grassmann space. Inner products

We assume that differentials of Grassmann coordinates $d\theta^a$ fulfill the Grassmann anticommuting relations [2, 3]

$$\{d\theta^a, d\theta^b\} = 0 \tag{2.18}$$

and we introduce a single integral over the whole interval of $d\theta^a$

$$\int d\theta^a = 0, \quad \int d\theta^a\theta^a = 1, a = 0, 1, 2, 3, 5, .., d, \tag{2.19}$$

and the multiple integral over d coordinates

$$\int d^d\theta\,\theta^0\theta^1\theta^2\theta^3\theta^4...\theta^d = 1, \tag{2.20}$$

with

$$d^d\theta := d\theta^d...d\theta^3 d\theta^2 d\theta^1 d\theta^0.$$

We define [2, 3] the inner product of two vectors $< \varphi|\theta >$ and $< \theta|\chi >$, with $< \varphi|\theta >=< \theta|\varphi >^*$ as follows:

$$< \varphi|\chi >= \int d^d\theta(\omega < \varphi|\theta >) < \theta|\chi >, \qquad (2.21)$$

with the weight function ω

$$\omega = \prod_{k=0,1,2,3,..,d} (\frac{\partial}{\partial\theta^k} + \theta^k), \qquad (2.21a)$$

which operates on the first function $< \varphi|\theta >$ only, and we define

$$(\alpha^{a_1 a_2...a_k}\theta^{a_1}\theta^{a_2}...\theta^{a_k})^+ = (\theta^{a_k}).....(\theta^{a_2})(\theta^{a_1})(\alpha^{a_1 a_2...a_k})^*. \qquad (2.21b)$$

According to the above definition of the inner product it follows that $\tilde{a}^{a+} = -\eta^{aa}\tilde{a}^a$ and $\tilde{\tilde{a}}^{a+} = -\eta^{aa}\tilde{\tilde{a}}^a$, $(\tilde{a}^a\tilde{a}^b)^+ = -\eta^{aa}\eta^{bb}\tilde{a}^a\tilde{a}^b$, and $(\tilde{\tilde{a}}^a\tilde{a}^b)^+ = -\eta^{aa}\eta^{bb}\tilde{\tilde{a}}^a\tilde{a}^b$. The generators of the Lorentz transformations (eqs.(2.12a) and (2.12b)) are self adjoint (if $a \neq 0$ and $b \neq 0$) or anti self adjoint (if $a = 0$ or $b = 0$) operators.

According to eqs.(2.7) and (2.12) we find

$$S^{ab} = -i(\theta^a\frac{\partial}{\partial\theta_b} - \theta^b\frac{\partial}{\partial\theta_a}), \qquad (2.22a)$$

$$\tilde{a}^a = (\frac{\partial}{\partial\theta_a} + \theta^a), \quad \tilde{\tilde{a}}^a = i(\frac{\partial}{\partial\theta_a} - \theta^a). \qquad (2.22b)$$

$$\tilde{S}^{ab} = \frac{-i}{2}(\frac{\partial}{\partial\theta_a}+\theta^a)(\frac{\partial}{\partial\theta_b}+\theta^b), \quad \tilde{\tilde{S}}^{ab} = \frac{i}{2}(\frac{\partial}{\partial\theta_a}-\theta^a)(\frac{\partial}{\partial\theta_b}-\theta^b), \; if a \neq b, \quad (2.22c)$$

The eigenvalue problem

To find eigenvectors of any operator A, we solve the eigenvalue problem

$$< \theta|\tilde{A}_i|\tilde{\varphi} >= \tilde{\alpha}_i < \theta|\tilde{\varphi} >, \quad < \theta|A_i|\varphi >= \alpha_i < \theta|\varphi >, \;\; i = \{1,r\}, \qquad (2.23)$$

where $\tilde{A}_i$ and A_i stand for r commuting operators of spinorial and vectorial character, respectively.

To solve equations (2.23) we express the operators in the coordinate representation (Eqs.(2.22)) and write the eigenvectors as polynomials of θ^a. We orthonormalize the vectors according to the inner product, defined in Eq.(2.21)

$$< {}^a\tilde{\varphi}_i|{}^b\tilde{\varphi}_j >= \delta^{ab}\delta_{ij}, \quad < {}^a\varphi_i|{}^b\varphi_j >= \delta^{ab}\delta_{ij}, \qquad (2.23a)$$

where index a distinguishes between vectors of different irreducible representations and index j between vectors of the same irreducible representation. Eq.(2.23a) determines the orthonormalization condition for spinorial and vectorial representations, respectively.

LORENTZ GROUPS AND SUBGROUPS

The algebra of the group $SO(1, d-1)$ *or* $SO(d)$ contains [1] n subalgebras defined by operators $\tau^{Ai}, A = 1, n; i = 1, n_A$, where n_A is the number of elements of each subalgebra, with the properties

$$[\tau^{Ai}, \tau^{Bj}] = i\delta^{AB} f^{Aijk}\tau^{Ak}, \qquad (3.1)$$

if operators τ^{Ai} can be expressed as linear superpositions of operators M^{ab}

$$\tau^{Ai} = c^{Ai}{}_{ab}M^{ab}, \quad c^{Ai}{}_{ab} = -c^{Ai}{}_{ba}, \quad A = 1, n, \quad i = 1, n_A, \quad a, b = 1, d. \quad (3.1a)$$

Here f^{Aijk} are structure constants of the (A) subgroup with n_A operators. According to the three kinds of operators M^{ab}, two of spinorial and one of vectorial character, there are three kinds of operators τ^{Ai} defining subalgebras of spinorial and vectorial character, respectively, those of spinorial types being expressed with either $\tilde{S}^{ab}$ or $\tilde{\tilde{S}}^{ab}$ and those of vectorial type being expressed by S^{ab}. All three kinds of operators are, according to Eq.(3.1), defined by the same coefficients $c^{Ai}{}_{ab}$ and the same structure constants f^{Aijk}. From Eq.(3.1) the following relations among constants $c^{Ai}{}_{ab}$ follow:

$$-4c^{Ai}{}_{ab}c^{Bjb}{}_{c} - \delta^{AB} f^{Aijk}c^{Ak}{}_{ac} = 0. \qquad (3.1b)$$

In the case when the algebra and the chosen subalgebras are isomorphic, that is if the number of generators of subalgebras is equal to $\frac{d(d-1)}{2}$, the inverse matrix e^{Aiab} to the matrix of coefficients $c^{Ai}{}_{ab}$ exists [1] $M^{ab} = \sum_{Ai} e^{Aiab}\tau^{Ai}$, with the properties $c^{Ai}{}_{ab}e^{Bjab} = \delta^{AB}\delta^{ij}$, $c^{Ai}{}_{cd}e^{Aiab} = \delta^a{}_c\delta^b{}_d - \delta^b{}_c\delta^a{}_d$.

When we look for coefficients $c^{Ai}{}_{ab}$ which express operators τ^{Ai}, forming a subalgebra $SU(n)$ of an algebra $SO(2n)$ in terms of M^{ab}, the procedure is rather simple [6, 2]. We find:

$$\tau^{Am} = -\frac{i}{2}(\tilde{\sigma}^{Am})_{jk}\{M^{(2j-1)(2k-1)} + M^{(2j)(2k)} + iM^{(2j)(2k-1)} - iM^{(2j-1)(2k)}\}. \qquad (3.2)$$

Here $(\tilde{\sigma}^{Am})_{jk}$ are the traceless matrices which form the algebra of $SU(n)$. One can easily prove that operators τ^{Am} fulfil the algebra of the group $SU(n)$ for any of three choices for operators $M^{ab} : S^{ab}, \tilde{S}^{ab}, \tilde{\tilde{S}}^{ab}$.

In reference [2] coefficients $c^{Ai}{}_{ab}$ for a few cases interesting for particle physics can be found. Of special interest is the group $SO(1, 14)$ with the

subgroups $SO(1,4)$ and $SO(10) \supset SU(3) \times SU(2) \times U(1)$ which enables the unification of spins and charges. As have we already said, the coefficients are the same for all three kinds of operators, two of spinorial and one of vectorial character. The representations, of course, depend on the operators M^{ab} (See Eqs.(2.22)). After solving the eigenvalue problem (Eqs.(2.23)) for the invariants of the subgroups, the representations can be presented as polynomials of coordinates $\theta^a, a = 0, 1, 2, 3, 5, .., 15$. The operators of spinorial character define the fundamental representations of the group and the subgroups, while the operators of vectorial character define the adjoint representations of the groups.

We shall comment on the representations in the last section.

LAGRANGE FUNCTION FOR FREE PARTICLES IN ORDINARY AND GRASSMANN SPACE AND CANONICAL QUANTIZATION

We present in this section the Lagrange function for a particle which lives in a d-dimensional ordinary space of commuting coordinates and in a d-dimansional Grassmann space of anticommuting coordinates $X^a \equiv \{x^a, \theta^a\}$ and has its geodesics parametrized by an ordinary Grassmann even parameter (τ) and a Grassmann odd parameter(ξ). We derive the Hamilton function and the corresponding Poisson brackets and perform the canonical quantization, which leads to the Dirac equation with operators, which are differential operators in ordinary and in Grassmann space.

$X^a = X^a(x^a, \theta^a, \tau, \xi)$ are called supercoordinates. We define the dynamics of a particle by choosing the action [1, 4]

$$I = \frac{1}{2} \int d\tau d\xi E E^i_A \partial_i X^a E^j_B \partial_j X^b \eta_{ab} \eta^{AB}, \tag{4.1}$$

where $\partial_i := (\partial_\tau, \overrightarrow{\partial}_\xi), \tau^i = (\tau, \xi)$, while E^i_A determines a metric on a two dimensional superspace τ^i , $E = det(E^i_A)$. We choose $\eta_{AA} = 0, \eta_{12} = 1 = \eta_{21}$, while η_{ab} is the Minkowski metric with the diagonal elements $(1, -1, -1, -1, ..., -1)$. The action is invariant under the Lorentz transformations of supercoordinates: $X'^a = L^A_b X^b$. (See Eq.(2.3)). Since a supermatrix E^i_A transforms as a vector in a two-dimensional superspace τ^i under general coordinate transformations of τ^i, $E^i_A \tau_i$ is invariant under such transformations and so is $d^2 \tau E$. The action (4.1) is therefore locally supersymmetric. The inverse matrix E^A_i is defined as follows: $E^i_A E^B_i = \delta^B_A$.

Taking into account that either x^a or θ^a depend on an ordinary time parameter τ and that $\xi^2 = 0$, the geodesics can be described as a polynomial of ξ as follows: $X^a = x^a + \varepsilon \xi \theta^a$. We choose ε^2 to be equal either to $+i$ or to $-i$ so that it defines two possible combinations of supercoordinates. Accordingly we also choose the metric $E^i_A : E^1_1 = 1, E^1_2 = -\varepsilon M, E^2_1 = \xi, E^2_2 = N - \varepsilon \xi M$, with N and M Grassmann even and odd parameters, respectively. We write $\dot{A} = \frac{d}{d\tau} A$, for any A.

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If we integrate the action(4.1) over the Grassmann odd coordinate $d\xi$, the action for a superparticle follows:

$$\int d\tau (\frac{1}{N}\dot{x}^a \dot{x}_a + \varepsilon^2 \dot{\theta}^a \theta_a - \frac{2\varepsilon^2 M}{N}\dot{x}^a \theta_a). \qquad (4.1a)$$

Defining the two momenta

$$p_a^\theta := \frac{\overrightarrow{\partial} L}{\partial \dot{\theta}^a} = \epsilon^2 \theta^a, \qquad (4.2a)$$

.

$$p_a := \frac{\partial L}{\partial \dot{x}^a} = \frac{2}{N}(\dot{x}_a - Mp^{\theta a}), \qquad (4.2b)$$

the two Euler-Lagrange equations follow:

$$\frac{dp^a}{d\tau} = 0, \quad \frac{dp^{\theta a}}{d\tau} = \varepsilon^2 \frac{M}{2} p^a. \qquad (4.3)$$

Variation of the action(4.1a) with respect to M and N gives the two constraints

$$\chi^1 := p^a a_a^\theta = 0, \chi^2 = p^a p_a = 0, \quad a_a^\theta := ip_a^\theta + \varepsilon^2 \theta_a, \qquad (4.4)$$

while $\chi^3{}_a := -p_a^\theta + \epsilon^2 \theta_a = 0$ (eq.(4.2a) is the third type of constraints of the action(4.1). For $\varepsilon^2 = -i$ we find, if using Eq.(2.7), that $a^\theta{}_a = \tilde{a}^a$, $\chi^3{}_a = \tilde{\tilde{a}}_a = 0$.

We find the generators of the Lorentz transformations for the action(4.1) to be (See Eq.(2.12))

$$M^{ab} = L^{ab} + S^{ab} \, , \ L^{ab} = x^a p^b - x^b p^a \, , \ S^{ab} = \theta^a p^{\theta b} - \theta^b p^{\theta a} = \tilde{S}^{ab} + \tilde{\tilde{S}}^{ab}, \ (4.5)$$

which show that parameters of the Lorentz transformations are the same in both spaces.

We define the Hamilton function:

$$H := \dot{x}^a p_a + \dot{\theta}^a p^\theta{}_a - L = \frac{1}{4}Np^a p_a + \frac{1}{2}Mp^a(\tilde{a}_a + i\tilde{\tilde{a}}_a) \qquad (4.6)$$

and the corresponding Poisson brackets

$$\{A, B\}_p = \frac{\partial A}{\partial x^a}\frac{\partial B}{\partial p_a} - \frac{\partial A}{\partial p_a}\frac{\partial B}{\partial x^a} + \frac{\overrightarrow{\partial A}}{\partial \theta^a}\frac{\overrightarrow{\partial B}}{\partial p_a^\theta} + \frac{\overrightarrow{\partial A}}{\partial p_a^\theta}\frac{\overrightarrow{\partial B}}{\partial \theta^a}, \qquad (4.7)$$

which have the properties of the generalized commutators presented in Eqs.(2.8-2.9).

If we take into account the constraint $\chi^3{}_a = \tilde{\tilde{a}}_a = 0$ in the Hamilton function (which just means that instead of H the Hamilton function $H + \sum_i \alpha^i \chi^i + \sum_a \alpha^3{}_a \chi^{3a}$ is taken, with parameters $\alpha^i, i = 1, 2$ and $\alpha^3{}_a = -\frac{M}{2}p_a, a = 0, 1, 2, 3, 5, .., d$ chosen on ssuch a way that the Poisson brackets of the three types of constraints

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with the new Hamilton function are equal to zero) and in all dynamical quantities, we find:

$$H = \frac{1}{4}Np^a p_a + \frac{1}{2}Mp^a \tilde{a}_a, \ \ \chi^1 = p^a p_a = 0, \ \ \chi^2 = p^a \tilde{a}_a = 0, \qquad (4.4a)$$

$$\dot{p}_a = \{p_a, H\}_P = 0, \dot{\tilde{a}}_a = \{\tilde{a}_a, H\}_P = iMp_a, \qquad (4.3a)$$

which agrees with the Euler Lagrange equations (4.3).

We further find

$$\dot{\chi}^i = \{H, \chi^i\}_P = 0, \ \ i = 1, 2, \ \ \dot{\chi}^3{}_a = \{H, \chi^3{}_a\}_P = 0, \ \ a = 0, 1, 2, 3, 5, ..., d, \qquad (4.3b)$$

which guarantees that the three constraints will not change with the time parameter τ and that $\dot{\tilde{M}}^{ab} = 0$, with $\tilde{M}^{ab} = L^{ab} + \tilde{S}^{ab}$, saying that $\tilde{M}^{ab}$ is the constant of motion.

The Dirac brackets, which can be obtained from the Poisson brackets of Eq.(4.7) by adding to these brackets on the right hand side a term $-\{A, \tilde{\tilde{a}}^c\}_P \cdot (-\frac{1}{2i}\eta_{ce}) \cdot \{\tilde{\tilde{a}}^e, B\}_P$, give for the dynamical quantities, which are observables, the same results as the Poisson bracket. This is true also for $\tilde{a}^a$, ($\{\tilde{a}^a, \tilde{a}^b\}_D = i\eta^{ab} = \{\tilde{a}^a, \tilde{a}^b\}_P$), which is the dynamical quantity but not an observable since its odd Grassmann character causes supersymmetric transformations. We also find that $\{\tilde{a}^a, \tilde{\tilde{a}}^b\}_D = 0 = \{\tilde{a}^a, \tilde{\tilde{a}}^b\}_P$. The Dirac brackets give different results only for the quantities θ^a and $p^{\theta a}$ and for $\tilde{\tilde{a}}^a$ among themselves: $\{\theta^a, p^{\theta b}\}_P = \eta^{ab}, \{\theta^a, p^{\theta b}\}_D = \frac{1}{2}\eta^{ab}, \{\tilde{\tilde{a}}^a, \tilde{\tilde{a}}^b\}_P = 2i\eta^{ab}, \{\tilde{\tilde{a}}^a, \tilde{\tilde{a}}^b\}_D = 0$. According to the above properties of the Poisson brackets, I suggest that in the quantization procedure the Poisson brackets (4.7) rather then the Dirac brackets are used, so that variables $\tilde{\tilde{a}}^a$, which are remooved from all dynamical quantities, stay as operators. Then $\tilde{a}^a$ and $\tilde{\tilde{a}}^a$ are expressible with θ^a and $p^{\theta a}$ (Eq.(2.7) and the algebra of linear operators introduced in Sect.2 (Eqs.(2.7) - (2.14)) can be used. We shall show, that suggested quantization procedure leads to the Dirac equation, which is the differential equation in ordinary and Grassmann space and has all desired properties.

In the proposed quantization procedure $-i\{A, B\}_p$ goes to either a commutator or to an anticommutator, according to the Poisson brackets (4.7). The operators $\theta^a, p^{\theta a}$ (in the coordinate representation they become $\theta^a \longrightarrow \theta^a$, $p^\theta_a \longrightarrow i\frac{\overrightarrow{\partial}}{\partial \theta^a}$) fulfill the Grassmann odd Heisenberg algebra, while the operators $\tilde{a}^a$ and $\tilde{\tilde{a}}^a$ fulfill the Clifford algebra(Eqs.(2.10))

The constraints (Eqs.(4.3)) lead to the Dirac like and the Klein-Gordon equations

$$p^a \tilde{a}_a |\tilde{\Psi}> = 0 \ , \ p^a p_a |\tilde{\Psi}> = 0, \ with \ p^a \tilde{a}_a p^b \tilde{a}_b = p^a p_a. \qquad (4.8)$$

Trying to solve the eigenvalue problem $\tilde{\tilde{a}}^a |\tilde{\Psi}> = 0, \ \ a = (0, 1, 2, 3, 5, ..., d)$, we find that no solution of this eigenvalue problem exists, which means that

the third constraint $\tilde{\tilde{a}}^a = 0$ can't be fulfilled in the operator form (although we take it into account in the operators for all dynamical variables in order that operator equations would agree with classical equations). We can only take it into account in the expectation value form

$$< \tilde{\Psi}|\tilde{\tilde{a}}^a|\tilde{\Psi} >= 0. \tag{4.9}$$

Since $\tilde{\tilde{a}}^a$ are Grassmann odd operators, they change monomials (Eq.(2.5)) of an Grassmann odd character into monomials of an Grassmann even character and opposite, which is the supersymmetry transformation. It means that Eq.(4.9) is fulfilled for momomials of either odd or even Grassmann character and that superpositions of the Grassmann odd and the Grassmann even monomials are not solutions for this system.

We can use the projector $P_\pm$ of Eq.(2.11) to project out of monomials either the Grassmann odd or the Grassmann even part. Since this projector commutes with the Hamilton function ($H = \frac{N}{4}p^a p_a + \frac{1}{2}\mu \ p^a \tilde{a}_b \tilde{a}_a, \ \ \{P_\pm, H\} = 0$), where we choose M to be proportional to $\tilde{a}_b$, for any $b \subset \{0,1,..,d\}$ and μ is a real parameter, it means that eigenfunctions of H have either an odd or an even Grassmann character. In order that in the second quantization procedure fields $|\tilde{\Psi} >$ would describe fermions, it is meaningful to accept in the fermion case Grassmann odd monomials only.

We further see that although the operators $\tilde{a}^a$ fulfil Clifford algebra, they cannot be recognized as the Dirac $\tilde{\gamma}^a$ operator, since they have an odd Grassmann character and therefore transform fermions into bosons, which is not the case with the Dirac γ^a matrices. We therefore recognize the generators of the Lorentz transformations $-2i\tilde{S}^{bm}, \ m = 0,1,2,3$, with $b = 5$ as the Dirac γ^m operators.

$$\tilde{\gamma}^m = -\tilde{a}^5 \tilde{a}^m = -2i\tilde{S}^{5m} \ , \ m = 0,1,2,3. \tag{4.10}$$

We choose the Dirac operators $\tilde{\gamma}^a$ in the way which in the case that $< \tilde{\psi}|p^5|\tilde{\psi} >= m$ and $< \tilde{\psi}|p^h|\tilde{\psi} >= 0$, for $h \in \{6,d\}$, enables to recognize the equation

$$(\tilde{\gamma}^m p_m - m)|\tilde{\psi} >= 0 \ , \ m = 0,1,2,3. \tag{4.11}$$

as the Dirac equation. The $m = 5$ is just the number of the previledged coordinate which determines the operators $\tilde{\gamma}^a$. Since $-2i\tilde{S}^{5m}$ appear as $\tilde{\gamma}^m$, $SO(1,4)$ rather than $SO(1,3)$ is needed to describe the spin degrees of freedom of fermionic fields.

It can be checked that in the four-dimensional subspace $\tilde{\gamma}^m$ fulfill the Clifford algebra $\{\tilde{\gamma}^m, \tilde{\gamma}^n\} = \eta^{mn}$, while $\tilde{S}^{mn} = -\frac{i}{4}[\tilde{\gamma}^m, \tilde{\gamma}^n]_-$.

We presented in Ref. [2] four Dirac four spinors (the polynomials of θ^a) which fulfil Eq.(4.11) if $m \neq 0$ and four Weyl four spinors which fulfill Eq.(4.11) if $m = 0$.

For large enough d not only do generators of Lorentz transformations (of a spinorial character) in Grassmann space define the spinorial degrees of freedom

of a particle field in the four dimensional subspace, but they also define the quantum numbers of these fields, which may be recognized as electromagnetic, weak and colour charges.

This certainly can be done for $d = 15$, since $SO(1, 14)$ has the subalgebra $SO(1, 4) \times SO(10)$, while $SO(10)$ has the subalgebra $SU(3) \times SU(2) \times U(1)$. In this case $\tilde{\tau}^{Ai}$ are linear superpositions of operators $\tilde{S}^{ab}$, $a, b \in \{6, d\}$ fulfilling the algebras as presented in Eqs.(3.1-3.3)

$$[\tilde{\tau}^{Ai}, \tilde{\tau}^{Bj}] = i\delta^{AB} f^{Aijk} \tilde{\tau}^{Ak}, \text{for } A, B, \tag{3.1a}$$

and defining the algebras of $SU(3), SU(2), U(1)$, while $SO(1, 4)$ remains to define the spinorial degrees of freedom in the four dimensional subspace. We find the fundamental representations of the corresponding Casimir operators as functions of θ^a determining isospin doublets, colour triplets and electromagnetic singlets [2].

PARTICLES IN GAUGE FIELDS

The dynamics of a point particle in gauge fields, the gravitational and the Yang-Mills fields, can be obtained by transforming vectors from a freely falling to an external coordinate system [5]. To do this, super vielbeins $\mathbf{e}^{ia}{}_\mu$ have to be introduced, which in our case depend on ordinary and on Grassmann coordinates, as well as on two types of parameters $\tau^i = (\tau, \xi)$. Since there are two kinds of derivatives ∂_i, there are two kinds of vielbeins [1, 2]. The index a refers to a freely falling coordinate system (a Lorentz index), the index μ refers to an external coordinate system (an Einstein index). Vielbeins with a Lorentz index smaller than five will determine ordinary gravitational fields, and those with a Lorentz index higher than four will define Yang-Mills fields. Spin connections appear in the theory as (a part of) Grassmann odd fields.

We write the transformation of vectors as follows

$$\partial_i X^a = \mathbf{e}^{ia}{}_\mu \partial_i X^\mu , \ \partial_i X^\mu = \mathbf{f}^{i\mu}{}_a \partial_i X^a , \ \partial_i = (\partial_\tau, \partial_\xi). \tag{5.1}$$

From eq.(5.1) it follows that

$$\mathbf{e}^{ia}{}_\mu \mathbf{f}^{i\mu}{}_b = \delta^a{}_b , \ \mathbf{f}^{i\mu}{}_a \mathbf{e}^{ia}{}_\nu = \delta^\mu{}_\nu. \tag{5.2}$$

Again we make a Taylor expansion of vielbeins with respect to ξ

$$\mathbf{e}^{ia}{}_\mu = e^{ia}{}_\mu + \varepsilon\xi\theta^b e^{ia}{}_{\mu b} , \ \mathbf{f}^{i\mu}{}_a = f^{i\mu}{}_a - \varepsilon\xi\theta^b f^{i\mu}{}_{ab} , \ i = 1, 2. \tag{5.3}$$

Both expansion coefficients again depend on ordinary and on Grassmann coordinates. Since $e^{ia}{}_\mu$ have an even Grassmann character they will describe the spin 2 part of a gravitational field. The coefficients $\varepsilon\theta^b e^{ia}{}_{\mu b}$ have an odd Grassmann character (ε is again the complex constant, we choose ε^2 equal to $-i$, so that $\tilde{a}^a = 0$). We shall see that they define the spin connections [1, 2].

From Eqs.(5.2) and (5.3) it follows that

$$e^{ia}{}_\mu f^{i\mu}{}_b = \delta^a{}_b \ , \ f^{i\mu}{}_a e^{ia}{}_\nu = \delta^\mu{}_\nu \ , \ e^{ia}{}_{\mu b} f^{i\mu}{}_c = e^{ia}{}_\mu f^{i\mu}{}_{cb} \ , \ i = 1, 2. \qquad (5.2a)$$

We find the metric tensor $\mathbf{g}^i_{\mu\nu} = \mathbf{e}^{ia}{}_\mu \mathbf{e}^i_{a\nu}$, $\mathbf{g}^{i\mu\nu} = \mathbf{f}^{i\mu}{}_a \mathbf{f}^{i\nu a}, i = 1, 2$, with an even Grassmann character and the properties $\mathbf{g}^{i\mu\sigma}\mathbf{g}^i_{\sigma\nu} = \delta^\mu{}_\nu = g^{i\mu\sigma} g^i_{\sigma\nu}$, with $g^i_{\mu\sigma} = e^{ia}{}_\mu e^i_{a\sigma}$.

It follows from Eq.(5.1) that vectors in a freely falling and in an external coordinate system are connected as follows: $\dot{x}^a = e^{1a}{}_\mu \dot{x}^\mu$, $\dot{x}^\mu = f^{1\mu}{}_a \dot{x}^a$, $\theta^a = e^{2a}{}_\mu \theta^\mu$, $\theta_\mu = f^{2\mu}{}_a \theta^a$, and $\dot\theta^a = e^{1a}{}_\mu \dot\theta^\mu + \theta^b e^{1a}{}_{\mu b} \dot{x}^\mu = (e^{2a}{}_\mu \theta^\mu)^\cdot = e^{2a}{}_{\nu,\mu_x} \dot{x}^\mu \theta^\nu + e^{2a}{}_\mu \dot\theta^\mu + \dot\theta^\mu \overrightarrow{e^{2a}}{}_{\nu,\mu_\theta} \theta^\nu$.

We use the notation $e^{2a}{}_{\nu,\mu^x} = \frac{\partial}{\partial x^\mu} e^{2a}{}_\nu, \overrightarrow{e^{2a}}{}_{\nu,\mu^\theta} = \frac{\overrightarrow{\partial}}{\partial\theta^\mu} e^{2a}{}_\nu$.

The above equations define the following relations among the fields:

$$e^{2a}{}_{\mu b} = 0 \ , \ \overrightarrow{e^{2a}}{}_{\nu,\mu^\theta} \theta^\nu = e^{1a}{}_\mu - e^{2a}{}_\mu \ , \ e^{1a}{}_{\mu b} = e^{2a}{}_{\nu,\mu^x} f^{2\nu}{}_b, \qquad (5.4)$$

which means that a point particle with a spin sees a spin connection $\theta^b e^{ia}{}_{\mu b}$ related to a vielbein $e^{2a}{}_\nu$.

Rewriting the action(4.1) in terms of an external coordinate system according to Eqs.(5.1), using the Taylor expansion of supercoordinates X^μ and superfields $\mathbf{e}^{ia}{}_\mu$ and integrating the action over the Grassmann odd parameter ξ, the action

$$I = \int d\tau \{ \frac{1}{N} g^1_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \ - \ \epsilon^2 \frac{2M}{N} \theta_a e^{1a}{}_\mu \dot{x}^\mu \ + \ \varepsilon^2 \frac{1}{2} (\dot\theta^\mu \theta_a - \theta_a \dot\theta^\mu) e^{1a}{}_\mu \ +$$

$$+ \ \varepsilon^2 \frac{1}{2} (\theta^b \theta_a - \theta_a \theta^b) e^{1a}{}_{\mu b} \dot{x}^\mu \},$$

$$(5.5)$$

defines the two momenta of the system

$$p_\mu = \frac{\partial L}{\partial \dot{x}^\mu} = p_{0\mu} + \frac{1}{2} \tilde{S}^{ab} e^1_{a\mu b}, \ \ p^\theta_\mu = -i\theta_a e^{1a}{}_\mu = -i(\theta_\mu + \overrightarrow{e}^{2a}{}_{\nu,\mu_\theta} e^2{}_{a\alpha} \theta^\nu \theta^\alpha), \qquad (5.6)$$

(where we made a choice of $\varepsilon^2 = -i$). Here $p_{0\mu}$ are the covariant (canonical) momenta of a particle. For $p^\theta_a = p^\theta_\mu f^{1\mu}{}_a$ it follows that p^θ_a is proportional to θ_a. Then $\tilde{a}_a = i(p^\theta_a - i\theta_a)$, while $\tilde{\tilde{a}}_a = 0$. We took this into account in the left hand expression of Eq.(5.6). We may further write

$$p_{0\mu} = p_\mu - \frac{1}{2} \tilde{S}^{ab} e^1_{a\mu b} = p_\mu - \frac{1}{2} \tilde{S}^{ab} \omega_{ab\mu} \ , \ \omega_{ab\mu} = \frac{1}{2} (e^1_{a\mu b} - e^1_{b\mu a}), \qquad (5.6a)$$

which is the usual expression for the covariant momenta in gauge gravitational fields [5]. One can find the two constraints

$$p_0^\mu p_{0\mu} = 0 = p_{0\mu} f^{1\mu}{}_a \tilde{a}^a. \tag{5.7}$$

To see how Yang-Mills fields enter into the theory, the Dirac equation (5.7) has to be rewritten in terms of fields which determine gravitation in the four dimensional subspace and of those fields which determine gravitation in higher dimensions, assuming that the coordinates of ordinary space with indices higher than four stay compacted to unmeasurably small dimensions. Since Grassmann space manifests itselfs through average values of observables only, compactification of a part of Grassmann space has no meaning. However, since parameters of Lorentz transformations in a freely falling coordinate system for both spaces have to be the same, no transformations to the fifth or higher coordinates may occur at measurable energies. Therefore, the four dimensional subspace of Grassmann space with the generators defining the Lorentz group $SO(1,3)$ is (almost) decomposed from the rest of the Grassmann space with the generators forming the (compact) group $SO(d-4)$, because of the decomposition of ordinary space. This is valide on the classical level only.

We shall assume the case in which only some components of fields differ from zero:

$$\begin{pmatrix} e^{im}{}_\alpha & 0 \\ \hline 0 & e^{ih}{}_\sigma \end{pmatrix}, \quad \alpha, m \in (0,3),\ \sigma, h \in (5,d),\ i \in (1,2), \tag{5.8}$$

while vielbeins $e^{im}{}_\alpha, e^{ik}{}_\sigma$ depend on $\theta^a and x^\alpha, \alpha \in 0,3$ only (since the dependence on other coordinates is not measurable). Accordingly we have (see Eq.(5.4)) only $\omega_{ab\alpha} \neq 0,$. We recognize, as in the freely falling coordinate system, that Grassmann coordinates with indices from 0 to 5 determine spins of fields, while Grassmann coordinates with indices higher than 5 determine charges of fields. We shall take expectation values of $p^h = 0$, $a \geq 6$ while we take $m = -p_5(e^{15}{}_5)^{-1}$. We find

$$\tilde{\gamma}^a f^{1\mu}{}_a p_{0\mu} = \tilde{\gamma}^m f^{1\alpha}{}_m (p_\alpha - \frac{1}{2}\tilde{S}^{mn}\omega_{mn\alpha} + A_\alpha) + m, \quad \text{where} A_\alpha = \sum_{A,i} \tilde{\tau}^{Ai} A^{Ai}_\alpha, \tag{5.9}$$

with $\sum_{A,i} \tilde{\tau}^{Ai} A^{Ai}_\alpha = \frac{1}{2}\tilde{S}^{hk}\omega_{hk\alpha},\ \ h,k = 6,7,8,..d.$

According to Eqs.(3.1) the Lie algebra of the Lorentz group $SO(d-5)$ contains the appropriate subalgebras for the desired charges if $\tilde{\tau}^{Ai}$ can be expressed as linear superpositions of operators $\tilde{S}^{ab}$, and f^{Aijk} are structure constants of the n subgroups A , each with n_A operators.

As we already stated in Section 2, this can certainly be done for $d = 15$, since $SO(1,14)$ has the subalgebras $SO(1,4) \times SO(10)$, while $SO(10)$ has the subalgebras $SU(3) \times SU(2) \times U(1)$.

We further find, if the subalgebras of operators $\tilde{\tau}^{Ai}$ are isomorphic to the algebra of $SO(d-5)$, that $A^{Ai}{}_{\alpha} = \frac{1}{2}\omega_{hk\alpha}d^{Aihk}$. Subalgebras of $SO(10)$ and $SU(3) \times SU(2) \times U(1)$ are not isomorphic. In this case there follow $\frac{10.(10-1)}{2}$ equations, which determine the same number of fields $\omega_{hk\alpha}$, for each $\alpha = 0,1,2,3$, in terms of $8 + 3 + 1$ fields $A^{Ai}{}_{\alpha}$. When these equations are fulfilled, the symmetry $SO(10)$ is broken to $SU(3) \times SU(2) \times U(1)$.

In eq.(5.9) the fields $\omega_{hk\alpha}$ determine all the Yang-Mills fields, including electromagnetic ones. The proposed unification differs from the Kaluza-Klein types of unification, since Yang-Mills fields are not determined by nondiagonal terms of vielbeins $e^{ih}{}_{\alpha}$. Instead they are determined with spin connections. In the proposed theory there is no difficulties either with the Planck mass of the electron (since the electron's charge is not determined by the momentum $p^{\sigma}, \sigma = 5$ but with the generators of Lorentz transformations in Grassmann space) or with transformation properties of gauge fields.

Torsion and curvature follow from the Poisson brackets $\{p_{0a}, p_{0b}\}_p$, with $p_{0a} = f^{1\mu}{}_a(p_\mu - \frac{1}{2}\tilde{S}^{cd}\omega_{cd\mu})$.

We find $\{p_{0a}, p_{0b}\}_p = -\frac{1}{2}S^{cd}R_{cdab} + p_{0c}T^c{}_{ab}$, $R_{cdab} = f^{1\mu}{}_{[a}f^{1\nu}{}_{b]}(\omega_{cd\nu,\mu}{}^x + \omega_c{}^e{}_\mu\omega_{ed\nu} + \overrightarrow{\omega}_{cd\mu,f^\theta}\theta^e\omega_e{}^f{}_\nu)$, $T^c{}_{ab} = e^{1c}{}_\mu(f^{1\nu}{}_{[b}f^{1\mu}{}_{a],\nu} + \omega_{e\nu}{}^d\theta^e f^{1\nu}{}_{[b}\overrightarrow{f^{1\mu}}{}_{a],d^\theta})$, with $A_{[a}B_{b]} = A_aB_b - A_bB_a$.

For $e^{im}{}_\alpha = \delta^m{}_\alpha$ one easily sees that Eq.(5.9) manifests the Dirac equation for a paricle with Yang-Mills charges in external fields.

While $\tilde{S}^{ab}$ determine the fundamental representations of the Lorentz group and therefore define $\tilde{S}^{mn}$, $m,n \in 0,3$ spins of fermionic fields and $\tilde{S}^{h,k}$, $h,k \in 6,d$ their Yang-Mills charges, determine accordingly $S^{a,b}$ adjoint representations of the Lorentz group and therefore define $S^{m,n}, m,n \in 0,3$ spins of bosonic fields and $S^{h,k}, h,k \in 6,d$ their charges.

CONCLUDING REMARKS

In this talk the theory in which space has d ordinary and d Grassmann coordinates was presented. Two kinds of generators of Lorentz tranformations in Grassmann space can be defined. The generators of spinorial character define the fundamental representations of the Lorentz group, the generators of the vectorial character define the adjoint representations of the Lorentz group. Both kinds of generators are the linear differential operators in Grassmann space. The Lorentz group $SO(1, d-1)$ contains for $d = 15$ as subgroups $SO(1,4), SU(3), SU(2)$ and $U(1)$. While $SO(1,4)$ defines spins of fermionic and bosonic fields, define $SU(3), SU(2)$ and $U(1)$ charges of both fields. Charges of fermionic fields belong to the fundamental representations, while charges of bosonic fields belong to the adjoint representations.

When looking for the representations of the operators $\tilde{S}^{mn}$, $m,n \in 0,3$ as polynomials of θ^a, $a \in 0,5$ and operators $\tilde{\tau}^{Ai}$ as polynomials of θ^h, $a \in 6,15$, we find representations of the group $SO(1,14)$ as outer products of the

representations of subgroups. The Grassmann odd polynomials, which are the Dirac four spinors, are triplets or singlets with respect to the colour charge, doublets or singlets with respect to the weak charge and may have hypercharge equal to $\pm\frac{1}{6}, \pm\frac{1}{3}, \pm\frac{2}{3}, \pm\frac{1}{2}, \pm1, 0$.

When looking for the representations of $SO(1, 14)$, as Grassmann even polynomials of θ^a, in terms of the subgroups $SO(1, 4), SU(3), SU(2), U(1)$, we find scalars and vectors, which are singlets, octets and multiplets with fourteen vectors with respect to the colour charge, triplets and singlets with respect to the weak charge and may have the hypercharge equal to 0 or to ±1. These representations are presented in Ref. [2].

We presented the Lagrange function for a particle living on a supergeodesics, with the momentum in the Grassmann space proportional to the Grassmann coordinate. In the quantization procedure the Dirac equation follows, with γ^a operators, which have the odd Grassmann character and are differential operators in Grassmann space with coordinates θ^a, $a \in 0, 5$. When transforming the Lagrange function from the freely fallyng to the external coordinate system, vielbeins and spin connections describe not only the gravitational field but also the Yang-Mills fields. Since the generators of the Lorentz transformations with indices higher then five determine charges of particles and spin connections again with indices higher then five describe the Yang-Mills fields (rather then vielbeins with one index smaller then five and another greater then three as in the Kaluza-Klein theories), the problems of the Kaluza-Klein theories (like a Planck mass of charged particles) do not occur.

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